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Abstract

An *orientation* of a graph G is a digraph D obtained from G by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v in D, denoted by $d_D^-(v)$, is the number of arcs with head v in D. An orientation D of G is *proper* if $d_D^-(u) \neq d_D^-(v)$, for all $uv \in E(G)$. The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum of the maximum indegree over all its proper orientations. It is well-known that $\overrightarrow{\chi}(G) \leq \Delta(G)$, for every graph G. In this paper, we first prove that $\overrightarrow{\chi}(G) \leq \left\lfloor \left(\Delta(G) + \sqrt{\Delta(G)}\right)/2 \right\rfloor + 1$ if G is a bipartite graph, and $\overrightarrow{\chi}(G) \leq 4$ if G is a tree. We then prove that deciding whether $\overrightarrow{\chi}(G) \leq \Delta(G) - 1$ is an $\mathcal{NP}$-complete problem. We also show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 2$, for planar *subcubic* graphs G. Moreover, we prove that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$, for planar bipartite graphs G with maximum degree 5.

Keywords: proper orientation, graph colouring, bipartite graph, hardness.

1. Introduction

In this paper, all graphs are *simple*, that is without loops and multiple edges. We follow standard terminology as used in [1].

An *orientation* D of a graph G is a digraph obtained from G by replacing each edge by just one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v in D, denoted by $d_D^-(v)$, is the number of arcs with head v in D. We use the notation $d^-(v)$ when the orientation D is clear from the context. The orientation D of G is *proper* if $d^-(u) \neq d^-(v)$, for all $uv \in E(G)$. An orientation with maximum indegree at most k is called a *k-orientation.* The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum integer k such that G admits a proper k-orientation. This graph parameter was introduced by Ahadi and Dehghan [2]. It is well-defined for any graph G since one can always obtain a proper $\Delta(G)$-orientation (see [2]). In other words, $\overrightarrow{\chi}(G) \leq \Delta(G)$. Note that every proper orientation of a graph G induces a proper vertex colouring of G. Thus, $\overrightarrow{\chi}(G) \geq \chi(G) - 1$. Hence, we have the following sequence of inequalities: $\omega(G) - 1 \leq \chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G)$.

These inequalities are best possible in the sense that, for a complete graph K, $\omega(K) - 1 = \chi(K) - 1 = \overrightarrow{\chi}(K) = \Delta(K)$. However, one might expect better upper bounds on some parameters by taking a convex

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combination of two others. Reed [3] showed that there exists $\epsilon_0 > 0$ such that $\chi(G) \le \epsilon_0 \cdot \omega(G) + (1-\epsilon_0)\Delta(G)$ for every graph G and conjectured the following.

Conjecture 1 (Reed [3])**.** For every graph G, $\chi(G) \le \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

If true, this conjecture would be tight. Johannson [4] settled Conjecture 1 for $\omega(G) = 2$ and $\Delta(G)$ sufficiently large.

Likewise, one may wonder if similar upper bounds might be derived for the proper orientation number.

Problem 1.

(a) Does there exist a positive ϵ_1 such that $\overrightarrow{\chi}(G) \le \epsilon_1 \cdot \omega(G) + (1-\epsilon_1)\Delta(G)$?

(b) Does there exist a positive ϵ_2 such that $\overrightarrow{\chi}(G) \le \epsilon_2 \cdot \chi(G) + (1-\epsilon_2)\Delta(G)$?

Observe that both questions are intimately related. Indeed if the answer to (a) is positive for ϵ_1, then the answer to (b) is also positive for ϵ_1. On the other hand, if the answer to (b) is positive for ϵ_2, then the answer to (a) is also positive for $\epsilon_1 = \epsilon_0 \cdot \epsilon_2$ by the above-mentioned result of Reed.

In Section 2, we answer Problem 1 positively in the case of bipartite graphs by showing that: if G is bipartite, then $\overrightarrow{\chi}(G) \le \left\lfloor \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rfloor + 1$. We also argue that this bound is tight for $\Delta(G) \in \{2,3\}$.

In Section 3, we prove that $\overrightarrow{\chi}(T) \le 4$, for every tree T. Moreover, we show that $\overrightarrow{\chi}(T) \le 3$ if $\Delta(T) \le 6$, and $\overrightarrow{\chi}(T) \le 2$ if $\Delta(T) \le 3$. We also argue that all these bounds are tight.

In Section 4, we study the computational complexity of computing the proper orientation number of a bipartite graph. In their seminal paper, Ahadi and Dehghan proved that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) = 2$ for planar graphs G. We first improve their reduction and show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \le 2$, for planar *subcubic* graphs G. Moreover, we prove that deciding whether $\overrightarrow{\chi}(G) \le \Delta(G) - 1$ is an $\mathcal{NP}$-complete problem for general graphs G. Finally, we show that it is also $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \le 3$ for planar bipartite graphs G with maximum degree 5.

Due to space limitation, we omit the proofs of these results.

2. General upper bound

Theorem 1. *Let G be a bipartite graph and let k be a positive integer. If $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$, then $\overrightarrow{\chi}(G) \le \Delta(G) - k$.*

Sketch of proof. In order to prove this theorem, we describe an algorithm (see Algorithm 1) that produces a proper $(\Delta(G)-k)$-orientation. Let $G = (X \cup Y, E)$ be a bipartite graph as in the statement of Theorem 1. The algorithm consists of two phases.

The first phase (lines 1 to 8 in Algorithm 1) produces an orientation, not necessarily proper, of the edges of G in such a way that the indegree of each vertex in X is at most k and the indegree of each vertex in Y is at most $\Delta(G) - k$. It proceeds as follows. We first orient all edges $xy \in E(G)$ from x to y, where $x \in X$ and $y \in Y$. Then we define k matchings as described subsequently.

Let $G_1 = G$, and let M_1 be a matching in G_1 that covers all vertices of maximum degree. For each $i \in \{2,\dots,k\}$, let G_i be the graph obtained from G_{i-1} by removing the edges in M_{i-1}, that is $G_i = G_{i-1} \setminus M_{i-1}$, and let M_i be a matching in G_i that covers all vertices of degree $\Delta(G_i)$. Such a M_i exists since it is well known that every bipartite graph H has a proper $\Delta(H)$-edge-colouring. Clearly, we have $\Delta(G_i) = \Delta(G_{i-1}) - 1$, for each $i \in \{2,3,\dots,k\}$. Let $M := \bigcup_{i=1}^{k} M_i$. Observe that if a vertex has degree $\Delta(G) - k + j$ in G, where $j \in \{1,2,\dots,k\}$, then it is incident to at least j edges in M. Hence, for all $j \in \{1,2,\dots,k\}$ and for each vertex y in Y of degree $\Delta(G) - k + j$ in G, we reverse the orientation of exactly j edges in M incident to y. This ends the first phase.

The second phase reverses the orientation of some edges in $E(G) \setminus M$, step by step, in order to obtain a $(\Delta(G)-k)$-orientation. This orientation is proper under the assumption of Theorem 1. $\square$

Algorithm 1: Proper Orientation of Bipartite Graphs

Input: Bipartite graph $G = (X \cup Y, E)$ and $k \in \mathbb{N}$ s.t. $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$.
Output: Proper $(\Delta(G) - k)$-orientation for G.

$G_1 \longleftarrow G$
Orient all edges in G from X to Y
for $i = 1, \dots, k$ **do**
  $M_i \longleftarrow$ matching of G_i saturating all vertices of degree $\Delta(G_i)$
  $G_{i+1} \longleftarrow G_i - M_i$
$M \longleftarrow \bigcup_{i=1}^{k} M_i$
foreach $y \in Y$ **do**
  reverse the orientation of $\max\{0; d_G(y) - \Delta(G) + k\}$ edges of M incident to y
$\widetilde{X} \longleftarrow X$
for $\ell = \Delta(G) - k - 1, \dots, 2$ **do**
  while $\exists x \in X$ *s.t.* $|N_{\leq \ell}(x)| \geq \ell - d^-(x)$ *and* $|N_{=\ell}(x)| \leq \ell - d^-(x)$ **do**
    $\widetilde{Y} \longleftarrow$ set of $\ell - d^-(x)$ vertices of highest indegree in $N_{\leq \ell}(x)$
    foreach $y \in \widetilde{Y}$ **do**
      Reverse the orientation of xy (i.e. re-orient xy towards x)
    $\widetilde{X} \longleftarrow \widetilde{X} \setminus \{x\}$

Theorem 2. *If G is a bipartite graph, then* $\overrightarrow{\chi}(G) \leq \left\lfloor \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rfloor + 1$.

Sketch of proof. By Theorem 1, for every $k \in \mathbb{N}$, if $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$, then $\overrightarrow{\chi}(G) \leq \Delta(G) - k$. In order to obtain a good upper bound for $\overrightarrow{\chi}(G)$, we must find the largest positive integer k such that the condition of Theorem 1 holds for a given graph G.

Solving the inequality for k, we obtain that $k < \frac{\Delta(G)-\sqrt{\Delta(G)}}{2}$. Since k is integer, we conclude that $\overrightarrow{\chi}(G) \leq \Delta(G) - \left\lceil \frac{\Delta(G)-\sqrt{\Delta(G)}}{2} \right\rceil + 1$, and the result follows. □

Note that if G is bipartite and $\Delta(G) \in \{2, 3, 4\}$, then the bound of Theorem 2 is equal to the trivial upper bound $\overrightarrow{\chi}(G) \leq \Delta(G)$. For $\Delta(G) = 1$ and $\Delta(G) = 2$, this bound is tight due to the paths with 2 and 4 vertices, respectively. In addition, there exists a bipartite graph G with $\Delta(G) = 3$ and $\overrightarrow{\chi}(G) = 3$.

3. Trees

Theorem 3. *If T is a tree, then the following statements hold:*

(1) if $\Delta(T) \leq 3$, then $\overrightarrow{\chi}(T) \leq 2$;
(2) if $\Delta(T) \leq 6$, then $\overrightarrow{\chi}(T) \leq 3$;
(3) $\overrightarrow{\chi}(T) \leq 4$.

Sketch of proof. We prove the three statements by using similar arguments. For $i \in \{1, 2, 3\}$, we consider a minimal counter-example M_i to statement (i) with respect to the number of vertices, and derive a contradiction that implies that no counter-example exists. Since M_i is a minimal counter-example, we have $\overrightarrow{\chi}(M_i) > i + 1$, but $\overrightarrow{\chi}(T) \leq i + 1$, for any proper subtree T of M_i. We use the latter fact to derive a proper $(i + 1)$-orientation of M_i, which contradicts $\overrightarrow{\chi}(M_i) > i + 1$. □

The three statements of the theorem are tight in the following sense: there is a tree with maximum degree 4 and proper orientation number 3, and a tree with maximum degree 7 and proper orientation number 4.

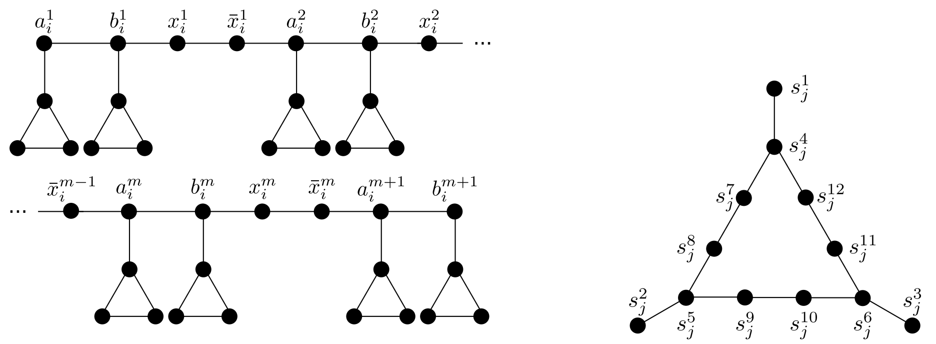
4. $\mathcal{NP}$-completeness

Ahadi and Dehgan [2] showed that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 2$ for planar graphs G by using a reduction from the PLANAR 3-SAT problem. We first improve this result by showing that it is $\mathcal{NP}$-complete to decide whether the proper orientation number of planar subcubic graphs is at most 2.

Theorem 4. *The following problem is $\mathcal{NP}$-complete:*
INPUT : *A planar graph G with $\Delta(G) = 3$ and $\delta(G) = 2$.*
QUESTION : $\overrightarrow{\chi}(G) \leq 2$*?*

Sketch of proof. We show a reduction from the problem of deciding whether a planar 3-SAT formula is satisfiable. It is known that the PLANAR 3-SAT problem is $\mathcal{NP}$-complete [5].

Let $\phi = (X, \mathcal{C})$ be an instance of this problem, where $X = \{x_1, \ldots, x_n\}$ is the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is the set of clauses. Using the variable and clause gadgets depicted in Figures 1(a) and 1(b), respectively, we construct a planar graph $G'(\phi)$ such that $\overrightarrow{\chi}(G'(\phi)) \leq 2$ if, and only if, ϕ is satisfiable. □



(a) Gadget associated to variable x_i.

(b) Gadget associated to clause C_j.

Figure 1: The variable and clause gadgets.

Recall that $\overrightarrow{\chi}(G) \leq \Delta(G)$, for any graph G. On the other hand, the following theorem shows that, for any integer $k \geq 3$, it is already $\mathcal{NP}$-complete to determine whether $\overrightarrow{\chi}(G) < k$, for graphs G with $\Delta(G) = k$.

Theorem 5. *Let k be an integer such that $k \geq 3$. The following problem is $\mathcal{NP}$-complete:*
INPUT : *A graph G with $\Delta(G) = k$ and $\delta(G) = k - 1$.*
QUESTION : $\overrightarrow{\chi}(G) \leq k - 1$*?*

Finally, we show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$, for planar bipartite graphs G.

Theorem 6. *The following problem is $\mathcal{NP}$-complete:*
INPUT : *A planar bipartite graph G with $\Delta(G) = 5$.*
QUESTION : $\overrightarrow{\chi}(G) \leq 3$*?*

Sketch of proof. We show a reduction from the problem of deciding whether a planar monotone 3-SAT formula is satisfiable. This problem was recently shown to be $\mathcal{NP}$-complete [6]. The idea of our reduction is roughly the same as in Theorems 4 and 5. □

References

[1] J. A. Bondy, U. S. R. Murty, Graph theory, Vol. 244 of Graduate Texts in Mathematics, Springer, New York, 2008.

[2] A. Ahadi, A. Dehghan, The complexity of the proper orientation number, Information Processing Letters 113 (19-21) (2013) 799–803.

[3] B. Reed, ω, δ, and χ, Journal of Graph Theory 27 (4) (1998) 177–212.

[4] A. Johansson, Asymptotic choice number for triangle free graphs, Technical Report 91–95, DIMACS (1996).

[5] D. Lichtenstein, Planar formulae and their uses, SIAM Journal on Computing 11 (2) (1982) 329–343.

[6] M. D. Berg, A. Khosravi, Optimal binary space partitions for segments in the plane, International Journal of Computational Geometry & Applications 22 (3) (2012) 187–205.